UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics M11B: Analysis 2

COURSE CODE	:	**MATHM11B**
UNIT VALUE	:	**0.50**
DATE	:	**09-MAY-01**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

01-C0978-3-160

TURN OVER

*All questions may be attempted but only marks obtained on the best **five** solutions will count. The use of an electronic calculator is **not** permitted in this examination.*

1. (i) Let f, g be functions differentiable at the point a. If $h(x) = f(x)g(x)$, show that $h'(a) = f'(a)g(a) + f(a)g'(a)$.

   (ii) Let f be a bounded differentiable function defined for all $x \in \mathbb{R}$. If $f(a) = \min f(x)$, show that $f'(a) = 0$.

   (iii) Let
$$\begin{aligned} f(x) &= x^3 & x \leqslant 1 \\ f(x) &= x^2 + x - 1 & x \geqslant 1 \end{aligned}.$$
   Show that $f'(1)$ exists but $f''(1)$ does not exist.

2. (i) Let f be a continuous strictly increasing function on a closed interval $[a, b]$. Show that f has a continuous strictly increasing inverse f^{-1} defined on $[f(a), f(b)]$. Show further that if $y \in (f(a), f(b))$, $y = f(x)$ say, with $f'(x)$ existing, $f'(x) \neq 0$, then f^{-1} is differentiable at y and
$$(f^{-1})'(y) = \frac{1}{f'(x)}.$$

   (ii) Let $f(x) = 1 - \cos x$, $x \in \left[0, \frac{\pi}{2}\right]$. Show that f has a continuous strictly increasing inverse f^{-1} defined on $[0, 1]$ and find $(f^{-1})'(y)$, where $0 < y < 1$.

3. (i) Define π and show that $2\sqrt{2} < \pi < 2\sqrt{3}$.

   (ii) Show that $\sin x > 0,\ 0 < x \leqslant 2$.

   (iii) Show that $\sin \frac{\pi}{2} = 1$.

   (Definitions based on geometric considerations will not be accepted).

4. (i) State and prove Rolle's theorem.

(ii) State and prove the Mean Value Theorem.

(iii) a) Find $\lim_{x\to 0} \dfrac{e^x - \cos x}{\tan x}$, b) $\lim_{x\to 0} \dfrac{(e^x - 1)\sin x}{\cos x - \cos^2 x}$,

c) $\lim_{x\to +\infty} x \log \dfrac{x+1}{x-1}$.

5. (i) Let f be a twice differentiable function on $\mathbb{R}$ whose second derivative f'' is continuous on $\mathbb{R}$. If $f'(a) = 0$ and $f''(a) > 0$ show that f has a local minimum at a.

(ii) Show that $\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{x^n}{n}, \quad -1 < x \leqslant 1.$

6. (i) Let f be an increasing function defined on $[0, 1]$. Show that f is Riemann integrable on $[0, 1]$.

(ii) If f and g are Riemann integrable functions on $[0, 1]$, show that, if $h(x) = f(x)g(x), x \in [0, 1]$, then h is Riemann integrable on $[0, 1]$.

(iii) If f is Riemann integrable on $[0, 1]$ and there exists $m > 0$ such that $f(x) \geqslant m,\ \forall x \in [0, 1]$, show that the function $1/f$ is Riemann integrable on $[0, 1]$.

7. (i) State and prove the Fundamental Theorem of Calculus.

(ii) Evaluate $\displaystyle\int_0^{\pi/4} \frac{dx}{3\cos^2 x + \sin^2 x}$.

(iii) Find $\displaystyle\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=0}^{n-1} k^2 \cos \frac{k\pi}{2n}$.